Properties of $E(Y)$

① If $Y = aX + b$ then

$$E(Y) = aE(X) + b \quad \left(\text{assuming } E(X) \text{ exists}\right).$$

② If you can find a constant a with $P(X \geq a) = 1$ then (naturally enough) $E(X) \geq a$; if b exists with $P(X \leq b) = 1$ then $E(X) \leq b$.

③ If $X_1, \ldots, X_n$ are n rvs, each with finite $E(X_i)$, then $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$,

④ and $E\left[\sum_{i=1}^{n} (aX_i + b)\right] = a\left(\sum_{i=1}^{n} E(X_i)\right) + b$.

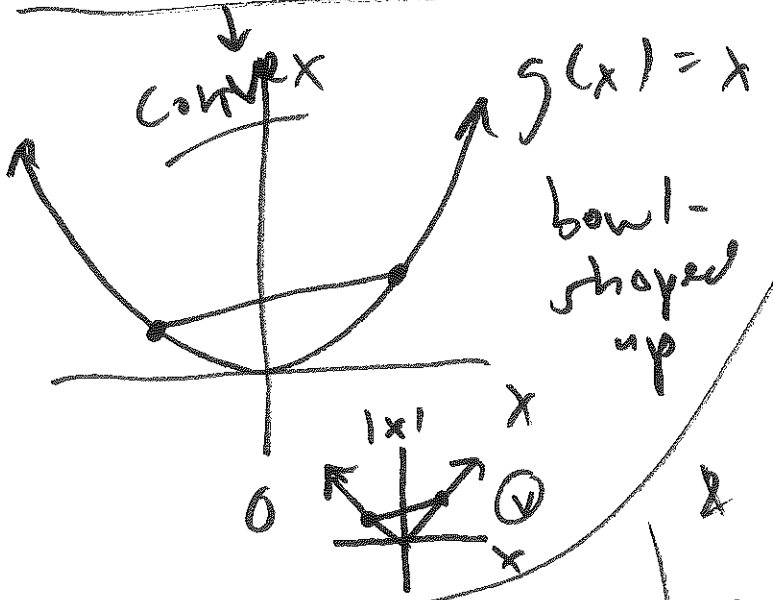
for all constants $(a_1, \ldots, a_n)$ and b.

Def. A function $g: \mathbb{R}^n \to \mathbb{R}$ (this means that $g(\underset{\sim}{x}) = z$) is convex

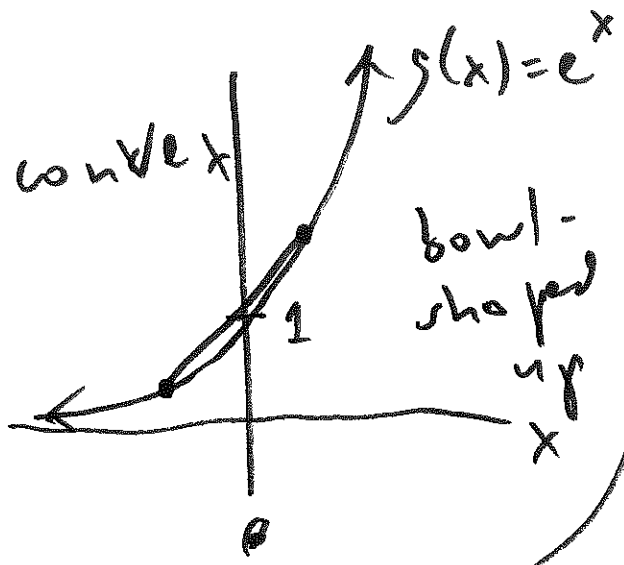
real #s

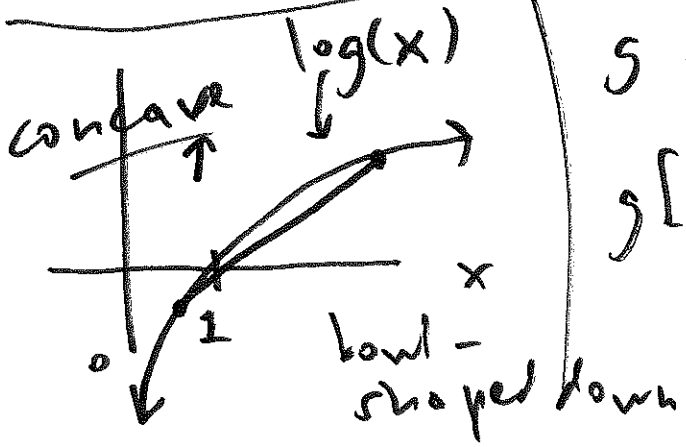
$(x_1, \ldots, x_n)$

if for every $0 < \alpha < 1$ and every $\underset{\sim}{x}$ and $\underset{\sim}{y}$, $g[\alpha \underset{\sim}{x} + (1-\alpha)\underset{\sim}{y}] \geq \alpha g(\underset{\sim}{x}) + (1-\alpha) g(\underset{\sim}{y})$



Graphical version of this: pick any two points on the function & connect them with a line segment; the function is convex if the line segment lies entirely above the function except at the endpoints.





g is <u>concave</u> if

$$g[\alpha \underset{\sim}{x} + (1-\alpha)\underset{\sim}{y}] \leq \alpha g(\underset{\sim}{x}) + (1-\alpha) g(\underset{\sim}{y})$$

Def. The expectation of a random vector $\underset{\sim}{X} = (X_1, \ldots, X_n)$ is $E(\underset{\sim}{X}) \triangleq \underbrace{[E(X_1), \ldots, E(X_n)]}_{n}$.

(a) g convex, $\underset{\sim}{X}$ random vector with finite $E(\underset{\sim}{X}) \rightarrow E[g(\underset{\sim}{X})] \geq g[E(\underset{\sim}{X})]$. Jensen's Inequality

(b) g concave $\rightarrow E[g(\underset{\sim}{X})] \leq g[E(\underset{\sim}{X})]$.

(attributed to Johan Jensen (1859 - 1925), Danish mathematician & engineer)

Application of 3

Suppose that $X_1, \ldots, X_n \overset{IID}{\sim} \text{Bernoulli}(p)$.

Then $E(X_i) = 0 \cdot \underset{P(X=0)}{(1-p)} + 1 \cdot \underset{P(X=1)}{p} = p$ and

$$E\Big(\underbrace{\sum_{i=1}^{n} X_i}_{\text{Binomial}(n,p)}\Big) = \sum_{i=1}^{n} E(X_i) = np = \text{mean of Binomial}(n,p)$$

Expectation of a product when the X_j are <u>independent</u>

$X_1, \ldots, X_n$ $\boxed{\text{independent}}$ rv, each with finite $E(X_i) \rightarrow$

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

Contrast this with a <u>sum</u>: $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$ whether the X_i are independent or not;

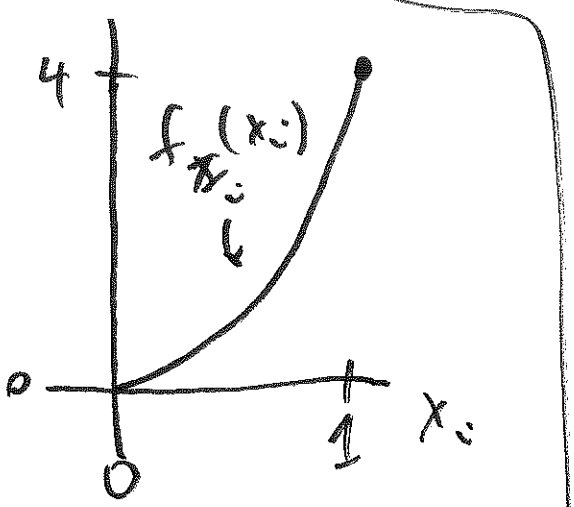
$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$ only when the X_j are <u>independent</u>.

$\boxed{\text{Example}}$ You have a Brita water filter that you use to improve the taste of Santa Cruz water. How much better would the filter do its job if you filtered the water twice instead of once?

X_1 = proportion of bad stuff removed in the 1st filtering

X_2 = proportion removed in 2nd filtering of what was left from 1st filtering

Reasonable to assume that X_1, X_2 are independent; suppose they're IID with common PDF $f_{X_i}(x_i) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & \text{else} \end{cases}$



(sensible shape)

Set Y = proportion of original bad stuff remaining after 2 filtrations $= (1-X_1)(1-X_2)$

Then $E(Y) = E[(1-X_1)(1-X_2)] \overset{\text{independence}}{=} E((1-X_1)) \cdot E((1-X_2))$

X_1, X_2 independent $\leftrightarrow (1-X_1), (1-X_2)$ independent too

$E(1-X_1) \overset{\text{identical distribution}}{=} E(1-X_2) \triangleq \mu$;

then $E(Y) = \mu^2$.

$\mu = E(1-X_i) = \int_0^1 (1-x_i)\,4x_i^3\,dx_i = 0.2$,

so 80% of bad stuff expected to be removed in 1st filtering; $E(Y) = \mu^2 = 0.04$, so expect only 4% of bad stuff to remain after 2 filterings.

6 (a) Suppose X is a discrete rv with possible values $0, 1, 2, \ldots$; then $E(X) = \sum_{n=1}^{\infty} P(X \geq n)$.

(b) If X is a continuous rv with possible values $(0, \infty)$, then $E(X) = \int_0^{\infty} [1 - F_X(x)]\,dx$, and CDF $F_X(x)$,

Example of 6 (a)

I throw a dart at a dartboard repeatedly, trying to get a bullseye (success). X = # of throw on which I first succeed.

(Ex. throws FFS → $X = 3$)

F = failure

S = success

Suppose that my success probability is constant across the throws and equals p, & throws are independent.

Then $E(X)$ should be inversely related to p:

the worse I am, the longer I expect the (1st bullseye) process to take; $E(X) = ?$

At least 1 throw always required so $P(X \geq 1) = 1$; for $n > 1$

(at least n tosses required) $\Leftrightarrow$ (none of the first (n-1) throws succeeded)

so $P(X \geq n) = (1-p)^{n-1}$ and geometric series

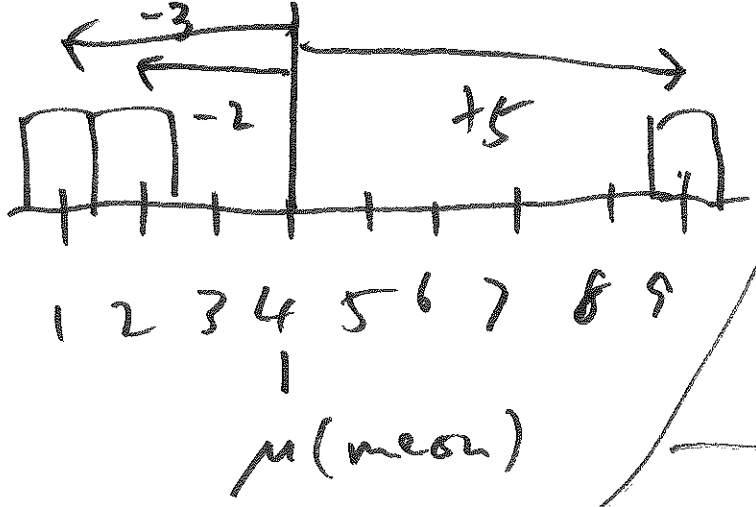
$$E(X) = \sum_{n=1}^{\infty} (1-p)^{n-1} = 1 + (1-p) + (1-p)^2 + \ldots$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}$$

(inverse relation ✓)

e.g. If I'm terrible ($p = .01$) I expect to succeed on the $\frac{1}{.01} = 100^{th}$ throw.

Variance and standard deviation



$\begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$

mean $4 = \mu$

X discrete rv, Uniform $\{1, 2, 9\}$; $E(X) = 4 = \mu$

$\boxed{\text{Q:}}$ How spread out is the dist. of X around its mean μ?

$(X - \mu) \sim$ Uniform $\{-3, -2, +5\}$ ← deviation from μ

Could try calculating $E(X - \mu)$, but this is 0 for any rv X, because of cancellation of ⊕ and ⊖ deviations; two different easy fixes: $E|X - \mu| \triangleq$ mean average absolute deviation (AAD) (MAD)

(Laplace)

or $E(X - \mu)^2 \triangleq$ variance of rv X.

(Gauss)

AAD not used much; variance used constantly.

Def X rv with finite mean $E(X) = \mu$;

<u>variance</u> of X = $V(X) = E\left[(X-\mu)^2\right]$.

Dfs we Var(X)

If $E(X) = \pm\infty$ or $E(X)$ doesn't exist, $V(X)$ doesn't exist.

One problem with variance

The units are wrong: if X is in \$, $V(X)$ is in \$^{2}.

Easy fix: <u>standard deviation</u> of X $\triangleq \sqrt{V(X)} \triangleq SD(X)$.

Consequences of these definitions

① $V(X) = E\left[(X-\mu)^2\right]$

$$= E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$$

$$= E(X^2) - \mu^2 = E(X^2) - [E(X)]^2.$$

This is a different way to compute the variance

so $V(X) = \left(\begin{array}{c}\text{expectation}\\ \text{of } X^2\end{array}\right) - \left(\begin{array}{c}\text{square of}\\ \text{expectation}\\ \text{of } X\end{array}\right)$.

Toy example $\begin{bmatrix}1\\2\\9\end{bmatrix}$ $X \sim \text{Uniform}\{1,2,9\}$

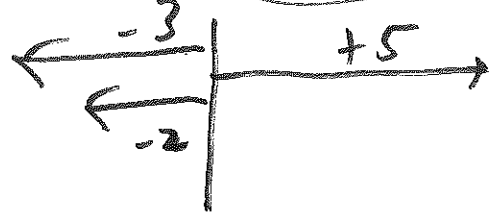
mean $\mu = 4$

$E(X-\mu)^2 = \frac{1}{3}(1-4)^2 + \frac{1}{3}(2-4)^2 + \frac{1}{3}(9-4)^2 \doteq 12.7 = V(X)$.

so

$SD(X) \doteq \sqrt{12.7} \doteq 3.6$

This is a reasonable summary of the length of the arrows



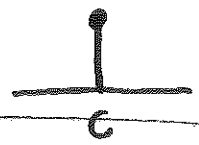
(2) For any rv X, $V(X) \geq 0$; if X is bounded, $V(X)$ exists & is finite.

This is a consequence of Jensen's Inequality:

$g(x) = x^2$ is convex so $E(X^2) \geq [E(X)]^2$,

i.e. $V(X) = E(X^2) - [E(X)]^2 \geq 0$.

③ $V(X) = 0 \overset{\text{iff}}{\longleftrightarrow} P(X = c) = 1$ for some constant c



(this is a trivial rv)

Notation In the same way that, by convention, $E(X) = \mu_X$, $V(X) \triangleq \sigma_X^2$ (lower-case sigma)

and $SD(X) \triangleq \sigma_X$

④ X rv, $Y = aX + b$

$\rightarrow V(Y) = a^2 V(X) = a^2 \sigma_X^2$ and

$SD(Y) = |a| \sigma_X$. (for any constants a, b)

Special cases $a = 0$: $V(X + c) = V(X)$

$SD(X + c) = SD(X)$

$V(aX) = a^2 V(X)$

$(b = 0)$ $SD(aX) = |a| SD(X)$

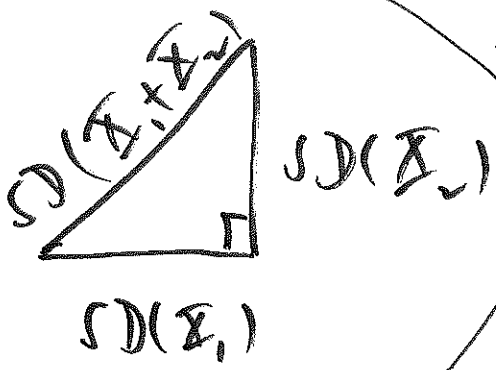
⑤ If $X_1, \ldots, X_n$ are <u>independent</u> rv with finite means, $V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i)$.

This is why the concept of variance (18) has endured even though the units of the variance are wrong: for independent rvs, variance is additive, whereas SD is not. (correct units)

Special case of (5):

X_1, X_2 independent $\Rightarrow V(X_1 + X_2) = V(X_1) + V(X_2)$

$= [SD(X_1)]^2 + SD(X_2^2)$



so

$SD(X_1 + X_2) = \sqrt{[SD(X_1)]^2 + [SD(X_2)]^2}$

i.e., SD grows like the hypotenuse of a right triangle.

Immediately, $\max\left\{\begin{matrix} SD(X_1), \\ SD(X_2) \end{matrix}\right\} < SD(X_1 + X_2) < SD(X_1) + SD(X_2)$

Consequence of 5: $X_1, \dots, X_n$ independent rv, $a_1, \dots, a_n, b$ constants $\rightarrow$

$$V\left[\left(\sum_{i=1}^{n} a_i X_i\right) + b\right] = \sum_{i=1}^{n} a_i^2 V(X_i).$$

Example: $X \sim \text{Binomial}(n, p)$; we already know that $E(X) = np$; what about $V(X)$ and $SD(X)$?

Let $S_i = \begin{cases} 1 & \text{if success on } i^{th} \text{ success/failure trial} \\ 0 & \text{else} \end{cases}$

for $(i = 1, \dots, n)$ and suppose as usual that $S_1, \dots, S_n$ are IID Bernoulli(p) — then $X = \sum_{i=1}^{n} S_i$ and we can work out its variance without difficulty.

$$V(X) = V\left(\sum_{i=1}^{n} S_i\right) \overset{\text{independence}}{=} \sum_{i=1}^{n} V(S_i)$$

so we need to work out the variance of a Bernoulli rv. We already know that $E(S_j) = p$, so if we use the formula $V(S_i) = E(S_i^2) - [E(S_i)]^2$ we're halfway there.

Bernoulli rvs are funny: $S_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{———— } (1-p) \end{cases}$

so $S_i^2 = \begin{cases} 1^2 = 1 & \text{with probability } p \\ 0^2 = 0 & \text{———— } (1-p) \end{cases}$

so $E(S_j^2) = E(S_j) = p$ and finally

$$V(S_j) = E(S_j^2) - [E(S_j)]^2 = p - p^2 = p(1-p)$$

and $V(X) = \sum_{i=1}^{n} V(S_i) = \sum_{i=1}^{n} p(1-p) = \boxed{np(1-p)}$

and $SD(X) = \sqrt{np(1-p)}$.

Example: T-S disease

X = (# T-S babies in family of $n=5$, both parents carriers so $p = P(\text{T-S baby}) = \frac{1}{4}$)

$\sim \text{Binomial}(n, p) = \text{Binomial}(5, \frac{1}{4})$

we already worked out that $E(X) = np$
$= 1.25$

Now $SD(X) = \sqrt{np(1-p)} = \sqrt{5(\frac{1}{4})(\frac{3}{4})}$
$\doteq 0.97$
$\doteq 1$

It's useful to summarize this by saying "The number of T-S babies this couple will have will be around 1.25, give or take about 1" $\leftarrow \sigma_X$ $\quad \mu_X$

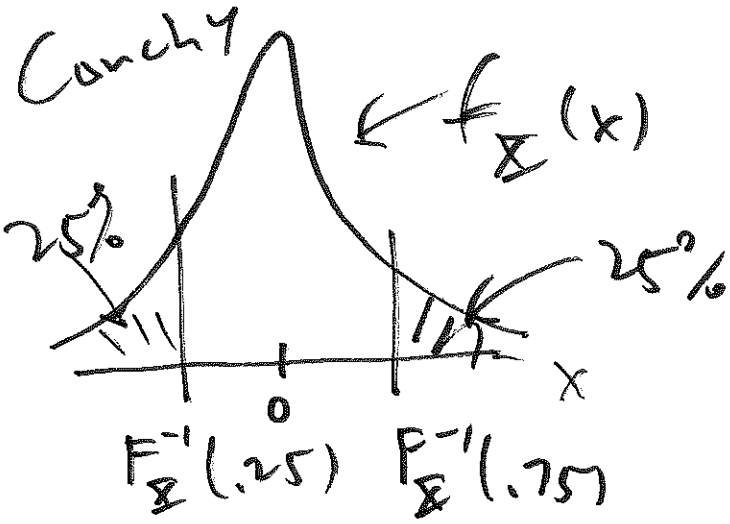
How do you measure the spread of a distribution if the variance doesn't exist?

Example $X \sim$ Cauchy

$$f_X(x) = \left\{ \frac{1}{\pi(1+x^2)} \text{ for all } -\infty < x < \infty \right\}$$

Earlier we saw that $E(X)$ doesn't exist, so clearly $V(X)$ doesn't exist either.

But we can use the idea of <u>quantiles</u> on <u>any</u> dist., whether its variance exists or not.



Earlier we defined the <u>interquartile range</u> (IQR) as

$$IQR = F_X^{-1}(.75) - F_X^{-1}(.25)$$

Cauchy CDF is $F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt$

(calculus) $= \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}$

Here $\tan^{-1}(x)$ (arctangent) is what's called the principal inverse of $\tan(x)$, ranging from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ as $-\infty < x < \infty$

Need to solve

$$F_X(x) = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi} = p \text{ for } x;$$

result is $x = F_X^{-1}(p) = \tan\left(\frac{p-\frac{1}{2}}{\pi}\right)$,

$= -\cot(p\pi)$

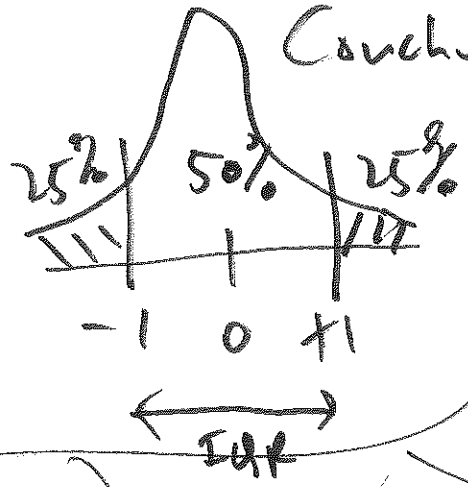
So the IQR for the Cauchy distribution is

$$\text{IQR} = F_X^{-1}\left(\tfrac{3}{4}\right) - F_X^{-1}\left(\tfrac{1}{4}\right)$$

$$= \tan\left(\tfrac{\pi}{4}\right) - \tan\left(-\tfrac{\pi}{4}\right)$$

$$= 2$$

Cauchy PDF



Moments of a rv

$$E(X) = E(X^1)$$

$$V(X) = E(X^2) - \left[E(X^1)\right]^2$$

With the usual mathematical impulse to generalize:

Def. X rv, k integer ≥ 1 $\rightarrow$

$$E(X^k) \triangleq \text{the } \underline{k^{th} \text{ moment}} \text{ of } X$$

of course $E(X^k)$ may not exist, and if it does it may be infinite, but the idea is still useful. You can show that

$$\left(\begin{array}{c}\text{kth moment} \\ \text{of } X \text{ exists}\end{array}\right) \longleftrightarrow E(|X|^k) < \infty$$

Consequences of the moment definition

① If $E(|X|^k) < \infty$ for some integer $k \geq 1$, then $E(|X|^j) < \infty$ for all integers $j < k$;

in other words, if the kth moment of X exists, so do the $(k-1)^{st}$, $(k-2)^{nd}$, ..., moments.

Definition

X rv with expectation $E(X) = \mu$, k integer ≥ 1 $\rightarrow$ $E\left[(X-\mu)^k\right]$ is called the <u>k^{th} central moment</u> of X or the <u>k^{th} moment of X around its mean</u>.

Clearly this idea generalizes the variance of $X = E\left[(X-\mu)^2\right]$

② $E\left[(X-\mu)^1\right] = E(X) - \mu = \mu - \mu = 0$,

i.e., every rv has 1st central moment 0.

③ If the dist. of X is symmetric around μ_X,

then $E\left[(X-\mu)^k\right] = 0$ for all odd

integers k for which $E\left[(X-\mu)^k\right]$ exists

This motivates a new definition:

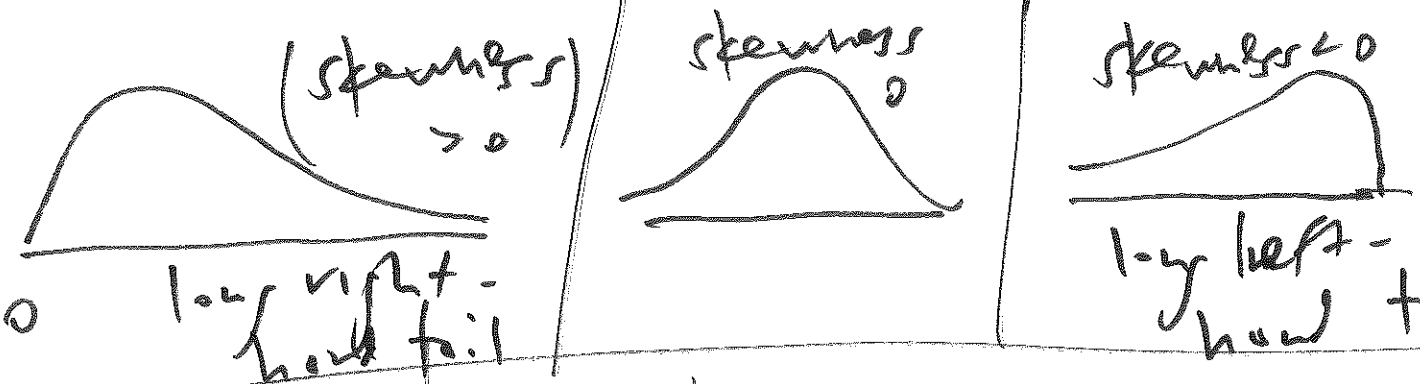
Def X rv with mean μ_X, SD σ_X;

if the third moment of X exists and

is finite, then $\underline{\text{skewness}(X)} \triangleq E\left(\frac{X-\mu_X}{\sigma_X}\right)^3$.

All symmetric distributions

with finite 3rd moment have skewness 0.



Moment generating functions

Def. X rv, t a real number

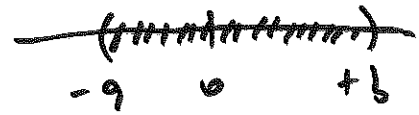
$\psi_X(t) \triangleq E(e^{tX})$ is called

psi the <u>moment generating function</u> of X (MGF)

The reason for this definition

Theorem X rv with MGF $\psi_X(t)$, finite for all values of t in an open interval $(-a, b)$ around 0 $\left(\begin{matrix} a>0 \\ b>0 \end{matrix}\right)$;

then for all integers $n > 0$,



$$E(X^n) = \frac{d^n}{dt^n}\psi_X(t)\Big|_{t=0}$$

$\leftarrow$ n^{th} derivative of ψ_X, evaluated at $t=0$.

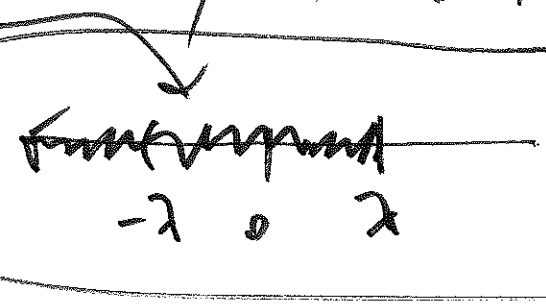
This is a handy theorem: if its premise is satisfied & the calculations are manageable, you get <u>all</u> the moments of X just by computing $\psi_X(t)$ and differentiating it over & over.

Example

$X \sim \text{Exponential}(\lambda)$	$(\lambda > 0)$ $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{else} \end{cases}$

$$\psi_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty e^{(t-\lambda)x}\, dx$$

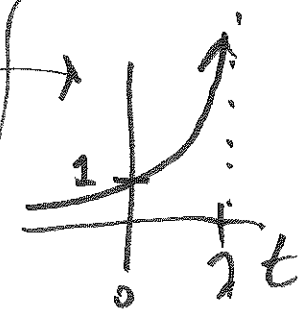
Now this integral is finite only if $t - \lambda < 0$, is for $t < \lambda$, but this means (since $\lambda > 0$)



that it's definitely finite in an open interval around 0 (e.g. $(-\lambda, \lambda)$).

So $\psi_X(t)$ exists for $t < \lambda$ and equals

$$\psi_X(t) = \lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \frac{\lambda}{\lambda - t}.$$



Now we just crank out the derivatives:

$$E(X^1) = \left(\frac{d}{dt}\frac{\lambda}{\lambda - t}\right)\Bigg|_{t=0} = \frac{1}{\lambda}$$

So $V(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \left[\frac{d^2}{dt^2}\left(\frac{\lambda}{\lambda - t}\right)\right]_{t=0} = \frac{2}{\lambda^2}$$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

and $SD(X) = \frac{1}{\lambda}$

$$E(X^3) = \left(\frac{d^3}{dt^3}\left(\frac{\lambda}{\lambda - t}\right)\right)_{t=0} = \frac{6}{\lambda^3}$$

positive skew (long right-hand tail)

$$E(X^4) = \left(\frac{d^4}{dt^4}\left(\frac{\lambda}{\lambda - t}\right)\right)_{t=0} = \frac{24}{\lambda^4}$$

Evidently $E(X^k) = \frac{k!}{\lambda^k}$. (12 Aug 16)