AMS 131 extra notes 1

Definition An <u>experiment</u> $\mathcal{E}$ is a data-generating process in which all possible outcomes can be listed before $\mathcal{E}$ is performed.

Definition An <u>event</u> E is a set of possible outcomes of an experiment $\mathcal{E}$.

Example: Tay-Sachs (T-S) disease

$\mathcal{E}$ = (the process by which the husband & wife end up with 5 children, each a T-S baby or not)

the E of interest is $E = \{\text{at least 1 T-S baby}\}$

Definition The <u>sample space</u> S is the <u>set</u> of all possible outcomes (elements) of an experiment $\mathcal{E}$.

Example: (T-5)

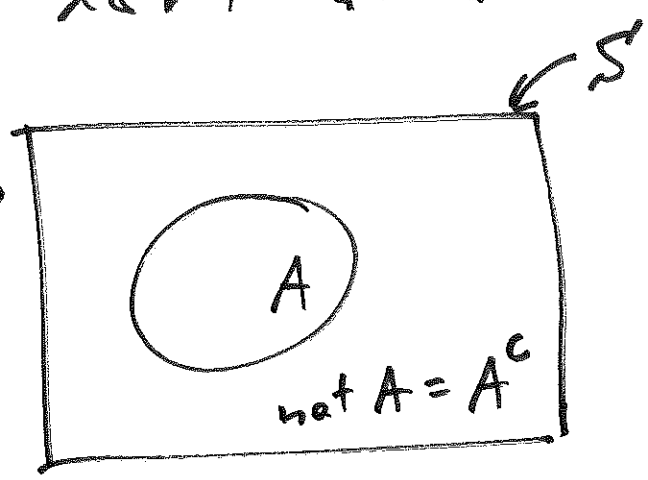
Let T = (T-5 baby) and N = (not T-5 baby)

N N N N N	Here $S = \{NNNNN, \ldots, TTTTT\}$
T N N N N	
N T N N N	Since there are 2 possibilities for each baby (T, N) and 5 babies, the number of elements in S is $2^5 = 32$.
N N T N N	
N N N T N	
N N N N T	
T T N N N (E)	
T N T N N	S is an example of a <u>product space</u>:
⋮	$\underbrace{\{T,N\} \times \{T,N\} \times \cdots \times \{T,N\}}_{5} = \{T,N\}^5$.
T T T T T	

Here $E = \{TNNNN, \ldots, TTTTT\}$.

Notation: Let us use s to stand for the individual outcomes (elements) of S.

The theory of probability we'll look at in this class was developed by Kolmogorov (1933) in an attempt to rigorize the hypothetical process of repeatedly throwing a dart at a Venn diagram (rectangle)



The rules of this dart-throwing were simple: ① the dart must land somewhere inside (or on the boundary of) the rectangle S, which

symbolically stands for the sample space, and 2 all the points where the dart might land in S are "equally likely" (as yet, an undefined primitive concept).

Definition The <u>complement</u> A^c of a set A contained in S is the set that contains all elements of S <u>not</u> in A

(You can see from the Venn diagram on p. 3 that the dart <u>has</u> to fall either in A or in A^c, which we could also call $\boxed{\text{not } A}$.)

Notation: $s \in S$ means that $\left\{\begin{matrix}\text{outcome}\\ \text{element}\end{matrix}\right\}$ s belongs to S (is an element of)

Definition A set A is contained in another set B (written $A \subset B$) if every element of A is also in B; can also say that B contains A ($B \supset A$).

(A is a subset of B)

Evidently, if A and B are events,

$A \subset B \overset{\text{(iff)}}{\longleftrightarrow}$ (if and only if) if A occurs then so does B

Consequences (theorem) If A, B, C are events

then (a) $A \subset B$ and $B \subset A \leftrightarrow A = B$

and (b) $A \subset B$ and $B \subset C \rightarrow A \subset C$.

Definition The cardinality of a set A (written $|A|$) is the number of distinct elements in A.

Example (Tay-Sachs) $|S| = 32$ (see p. 2)

Definition The set of all subsets of a given set S is called the <u>power set</u> of S, denoted by 2^S; this notation was chosen because, if $|S| = n$, then $|2^S| = 2^n$ (in other words, if S has n distinct elements then there are 2^n distinct subsets of S.

Definition It's convenient to have a symbol for the set that has no elements in it: $\emptyset$, the <u>empty set</u>.

Example If $S = \{a, b, c\}$ then

$|S| = 3$ and the power set has $2^3 = 8$ sets in it.

$\emptyset$	1
$\{a\}$	
$\{b\}$	3
$\{c\}$	
$\{a, b\}$	
$\{a, c\}$	3
$\{b, c\}$	
$\{a, b, c\} = S$	1

Given any set S (sample space), Kolmogorov (1933) wanted to be able to define probabilities in a logically-internally-consistent manner (in other words, free from contradictions or paradoxes) to <u>all</u> of the sets in 2^S.

		1		
	1		1	
1		2		1
1	3		3	1

Pascal's triangle

If $|S|$ is <u>finite</u>, it turns out that nothing nasty can happen.

But if $|S|$ is <u>infinite</u>, nasty things can unfortunately happen.

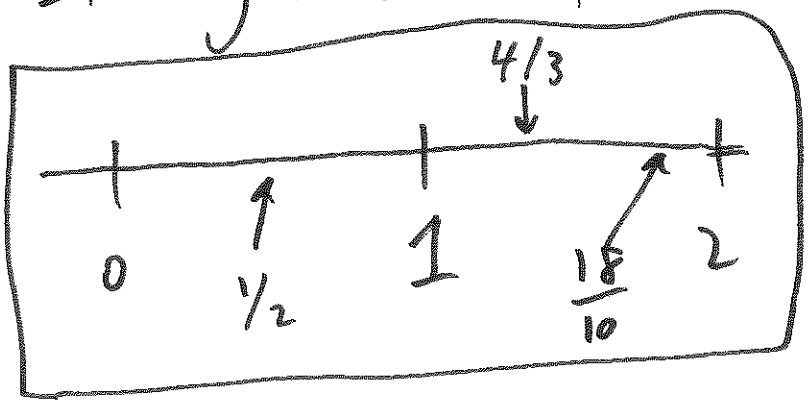
Definition

A set with an infinite number of distinct elements is called an <u>infinite set</u>.

Definition If the elements of an infinite set A can be placed in 1-to-1 correspondence with the positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$, A is said to be <u>countably infinite</u>.

Example The <u>rational numbers</u> are those real numbers that can be expressed as ratios of integers (ex. $\frac{1}{2}, \frac{14}{13}, -\frac{89}{212}, \ldots$)

It might seem that there are a lot more



rational numbers than integers, but Cantor (1878) showed that the rational numbers are <u>countable</u>. He also showed something even more surprising: the number of distinct values on the real number line is an <u>order of infinity greater</u> than the number of integers or <u>rationals</u>.

Definition An infinite set that is not countable is called <u>uncountable</u>.

Example $\mathbb{N} = \{1, 2, 3, \ldots\}$ is countable, but $\mathbb{R} = \{\text{all real numbers}\}$ is uncountable.

The mathematical foundation Kolmogorov chose for his development of probability theory is a part of mathematics called <u>measure theory</u>: an attempt to make rigorous the informal concepts of <u>length</u>, <u>area</u> and <u>volume</u> introduced by ancient Greek mathematicians including Euclid (about 2,300 years ago) and Pythagoras (about 2,500 years ago). However, in the early 1900s people discovered that infinity is a weird thing when you try to make an idea like volume of a sphere in 3-dimensional space rigorous.

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